

Last time: - Stated Sylow Thms
- Used them to classify gps of order pq

Normalizers

G a gp, $H \leq G$ a subgp.

$$N(H) = \{g \in G : gHg^{-1} = H\}$$

This is called the normalizer of H .

Have: $|G| = |N(H)| \cdot (\# \text{ of subgps conj. to } H)$

Propⁿ $H \leq G$ a subgp, $N = N(H)$ = the normalizer of H .

a) $H \leq N$ is normal

b) H is a normal subgp of $G \iff N = G$.

c) $|H|$ divides $|N|$ and $|N|$ divides $|G|$.

Suppose $G \leq S$. Then $G \leq$ subsets of S .

Lemma G fto gp, $S =$ subsets of G . $G \leq S$ by left mult.

Suppose $U \leq G$, denote $[U] \in S$ the corresponding cll.

Then $|\text{Stab}_G([U])|$ divides $|U|$ and $|G|$.

Pf: Set $H = \text{Stab}_G([U])$. Then $|H|$ divides $|G|$.

Pf: Let $H = \text{Stab}_G(U)$. Then $|H|$ divides $|G|$.

$$\Rightarrow H \leq U \leq G$$

This action partitions U into right H cosets, Hu

$\Rightarrow |U|$ is a multiple of $|H|$, b/c each coset has size $|H|$.

Lemma Suppose $n = p^e m$, $e > 0$ and $p \nmid m$. Then

$\binom{n}{p^e}$ is not divisible by p .

Pf: $(x+y)^n = \sum_{0 \leq k \leq n} \binom{n}{k} x^{n-k} y^k$ by Binomial Thm

$\Rightarrow \binom{n}{p^e}$ is the coeff of $x^{n-p^e} y^{p^e}$ in this expansion.

Claim: $(u+v)^p \equiv u^p + v^p \pmod{p}$

I.e., if u, v are indeterminates, then

$(u+v)^p - u^p - v^p$ is $p \times$ (integer combination of $u^A v^B$'s)

Pf: $(u+v)^p = u^p + v^p + \sum_{1 \leq j \leq p-1} \binom{p}{j} u^{p-j} v^j$

But $\binom{p}{j} = \frac{p!}{j!(p-j)!}$ is div. by p if $1 \leq j \leq p-1$.

Consequence $(u+v)^{p^2} \equiv (u^p + v^p)^p \equiv u^{p^2} + v^{p^2} \pmod{p}$

Consequence $(u+v)^{p^e} \equiv (u^p + v^p)^p \equiv u^{p^2} + v^{p^2} \pmod{p}$
 $\Rightarrow (u+v)^{p^e} \equiv u^{p^e} + v^{p^e} \pmod{p}.$

$$(x+y)^n = (x+y)^{p^e m} = ((x+y)^{p^e})^m \equiv (x^{p^e} + y^{p^e})^m \pmod{p}$$

$$\equiv x^{p^e} + \underbrace{m x^{p^e(m-1)} y^{p^e}}_{x^{n-p^e} y^{p^e}} + \dots \pmod{p}$$

$$\binom{n}{p^e} \equiv m \pmod{p} \Rightarrow p \nmid \binom{n}{p^e}. \quad \square$$

Prf of Thm 1 (Existence of Sylow p subgps)

G finite gp, $|G| = p^e m$, $e > 0$ $p \nmid m$.

S : subsets of G of size p^e .

G acts on S by left multiplication.

Now: $|S| = \binom{n}{p^e} = \text{not a multiple of } p.$

$$|S| = \sum_{\text{orbits } \mathcal{O}} |\mathcal{O}|$$

$\Rightarrow \exists$ orbit $\mathcal{O}_{[u]}$ with order not div. by p
 $H = \text{Stab}_G[u].$

$$H = \text{Stab}_G(u).$$

By lemma above, $|H| \mid |U| = p^e$.

But $|\mathcal{O}_G(u)| = \frac{|G|}{|H|}$ is not div by p .

$$\Rightarrow |H| = p^e.$$

Pf of Thm 2

Recall: a) All Sylow p subgps are conjugate
 b) Any p subgp K of G is contained in some Sylow p subgp.

a) and b) follow from: Given a p -subgp K of G , and a Sylow p subgp H of G , $\exists$ a conjugate H' of H that contains K .

$\mathcal{C} = G/H$ the cosets

Then $|\mathcal{C}| = \frac{|G|}{|H|} = m$ is not div by p .

$K \subseteq \mathcal{C}$ by restriction of G action.

By the fixed pt thm, $\exists c \in \mathcal{C}$ with $k \cdot c = c \ \forall k \in K$.

Therefore, $K \subseteq \text{Stab}_G(c) = \text{Stab}_G(gH) = gHg^{-1} = H' \quad \square$

Pf of Thm 3

Setup: $|G| = p^e m$, $p \nmid m$, $e > 0$. Let s denote the # of Sylow p subgps

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Then: $s \mid m$ and $s \equiv 1 \pmod{p}$.

S = set of Sylow p subgrps. $G \curvearrowright S$ by conjugation

Then: $|S| = s$ and action is transitive.

Consider H , a Sylow p subgp. $\leadsto [H] \in S$

The stabilizer of $[H]$ is $\text{Stab}_G [H] = N(H)$, the normalizer of H

$$s = |S| = \frac{|G|}{|N(H)|} = \frac{\frac{|G|}{|H|}}{\frac{|N(H)|}{|H|}} = \frac{m}{\frac{|N(H)|}{|H|}} \Rightarrow s \text{ divides } m.$$

$H \curvearrowright S$ by restriction. The orbit of $[H]$ is just $[H]$

Claim: All other orbits are nontrivial, and thus multiples of p .

Claim follows from H does not fix any other H' .

To see this, suppose H did fix $[H']$

Then $H \leq N(H') \Rightarrow H, H'$ are Sylow p subgrps of $N(H')$

• On the one hand, $\Rightarrow H + H'$ are conj in $N(H')$

• On the other hand, $H' \leq N(H')$ is normal.

$$\Rightarrow H = H'$$

□